Applied Algebra

George W. Myers

In our study of the equation we have already had frequent need for the resolution of numbers into factors. Many type forms of algebraic expressions will be met repeatedly in our later work with equations, and in many cases a knowledge of their factors will be of the greatest advantage. A degree of skill in the detection of factors is the surest possible guarantee of pleasurable and rapid progress in all advanced mathematical work. An intimate acquaintance with the difficulties of hundreds of students of applied mathematics convinces the writer that an imperfect understanding of the principles of factoring is one of the two or three most fruitful sources of trouble in all algebra. The reason for this is, however, not to be sought in the difficulty of the subject; but rather in the incessant recurrence of the necessity for a few ideas which are badly confused in the student's mind.

To illustrate the great advantage of the use of factoring a few exercises are given here.

Exercises

(1) Verbal statement. The difference of two numbers is 1, and the difference of their squares is 9; find the numbers.

Symbolic statement. $x-y=1$ and $x^2-y^2=9$; required x and y.

(2) Verbal statement. The sum of the reciprocals of two numbers and also the difference of the reciprocals of their squares is 5; find the numbers.

Symbolic statement. $1/x+1/y=5$ and $1/x^2-1/y^2=5$; find x and y.

(3) Verbal statement. The sum of two numbers is 5 and the sum of their cubes is 35; find the numbers.

Symbolic statement. $x+y=5$ and $x^3+y^3=35$; find x and y.

(4) Verbal statement. The sum of the cubes of two numbers is 126 and the sum of their squares diminished by their product is 21; find the numbers.

Symbolic statement. $x^3+y^3=126$ and $x^2-xy+y^2=21$; find x and y.

(5) Verbal statement. The difference of the 5th powers of two numbers is 242 and the difference of the numbers is 2; find the numbers.

Symbolic statement. $x^5-y^5=242$ and $x-y=2$; find x and y.

(6) $\frac{5x+1}{3}+\frac{19x+7}{9}-\frac{3x-1}{2}=\frac{7x-1}{6}$; find x.

(7) $\frac{2x+1}{2x-1}-\frac{8}{4x^2-1}=\frac{2x-1}{2x+1}$; find x.

(8) $\frac{4}{x+2}+\frac{7}{x+3}-\frac{37}{x^2+5x+6}=0$; find x.

(9) $\frac{x}{a-b}-\frac{5a}{a+b}=\frac{2bx}{a^2-b^2}$; find x.

(10) $\frac{(a^2+2ab+b^2)}{a+b}x+\frac{3a^2-3b^2}{a-b}x-\frac{2a^3+2b^3}{a^2-ab+b^2}x=\frac{ac+bc-ad-bd}{c-d}$; find x.

(11) $\frac{x^3-y^3}{x^2-xy+y^2}\times\frac{x^3+y^3}{x^2+xy+y^2}\times\left(1+\frac{y}{x-y}\right)=x+y$; find x.

Remark I.—Require the student to supply the verbal statements for 6, 7, 8, and 9. In practical life the problems to be solved are usually given in their verbal form, and much attention should be given to training the student to pass easily from the verbal to the symbolic form of statement and *vice versa*.

Remark II.—It is intended that the teacher, not the pupil, shall solve these problems to illustrate their simplicity by the aid of factoring.

Attention should be called to the fact that the first step in the solution of all of the first five exercises is to divide the equations, member by member, and that this division, by the aid of factoring, can be performed by *inspection*.

Factoring. The subject of factoring may be developed from three simple principles:

Definition: A polynomial is an algebraic expression containing two or more terms.

Principle I.—A factor of every term of any polynomial is a factor of the polynomial.

Illustration: $12+27-9=3\times4+3\times9-3\times3=3(4+9-3)$; $ax+ab-ac=a(x+b-c)$, etc.

Prove this principle.

Principle II.—The difference of any two numbers is a factor of the difference of any two positive integral powers of the numbers.

Illustration: $a-b$, $c-d$, $x-y$, $a-z$, $y-3$, are factors respectively of a^3-b^3, c^4-d^4, x^2-y^2, a^5-z^5, y^4-81.

Prove this principle.

Principle III.—If an algebraic expression becomes 0 when any number in it is replaced by another, the difference between the two numbers is a factor of the expression.

Illustration: If, in $7^3-3\times7^2+4\times7-4\ (=220)$, seven is replaced by 2, we have $2^3-3\times2^2+4\times2-4=0$. Hence $7-2=5$ is a factor of 220. If in $a^2-2ab+b^2$, a is replaced by b, the polynomial becomes 0. Hence $a-b$ is a factor of it.

Prove this principle.

Show that all the typical forms such as $a^2+2ab+b^2$, a^2-b^2, x^2+ax+b, etc., as well as the troublesome forms, x^n-y^n, x^n+y^n, come under this principle III.

Fractions.

Exercises 6 to 11 above are fractional in form. Before the value of the unknown number can be found the equation must be cleared, or freed, of the fractions. Most equations met with in practice involve fractions. In mechanics, the problems to be solved usually require as a preliminary step the differentiation, or integration, of some expression, and then the reduction and simplification of the results through the principles underlying fractions in algebra. The preliminary step is usually performed by the student with but little difficulty; the simplifying of the fractional and radical forms is the source of the greatest confusion. Some of the principles and notions which constitute the groundwork of the mechanics of materials are given here to show the practical importance of knowing how to treat equations which contain fractions. Sufficient explanation of the meaning of the forms used will be given to make the student feel their significance.

A study of *involution* and *evolution* will be made before factoring is completed.

Mechanics of Materials.

In designing beams the following equation is fundamental:

$$S=\frac{Mc}{I},$$

where S denotes the unit-stress in the fiber, M, the maximum statical moment, c, the distance from the remotest fiber of the section of the beam to the neutral plane, and I is the moment of inertia of this section.

I and c are here tabulated for beams of various forms and dimensions of cross-section.

Cross-section.	Dimensions.	Value of I.	Value of c.	Remarks.
Rectangle	Base, b; depth, d.	$\frac{bd^3}{12}$	$\frac{d}{2}$	----------
Circle	Diameter, d.	$\frac{\pi d^4}{64}$	$\frac{d}{2}$	Circle solid.
Triangle	Base, b; depth, d.	$\frac{bd^3}{36}$	$\frac{2}{3}d$	----------
Ellipse	Axes, a and b.	$\frac{4ab^3}{64}$	$\frac{b}{2}$	b vertical.
Square	Side, d.	$\frac{d^4}{12}$	$d\sqrt{\frac{1}{2}}$	Diagonal vertical.
I-beam	Base, b; depth, d.	$\frac{bd^3-(b-t')(d-2t)^3}{12}$	$\frac{1}{2}d$	Flange, t; web, t'.
⊥-beam	Base, b; depth, d; area, A.	$\frac{bd^3-(b-t')(d-t)^3}{3}-Ac^2$	$\frac{\frac{1}{2}t'd^2+t(b-t')(d-\frac{1}{2}t)}{t'd+t(b-t')}$	Flange, t; web, t'.
Circular	Diameters, d and d^1.	$\frac{\pi(d^4-d'^4)}{64}$	$\frac{d}{2}$	Hollow.

To see the use of the foregoing, design a hollow circular wrought-iron beam of 12′ span to carry a uniform load of 320 lbs. per lineal foot, thickness of metal to be ¾″.

Here we have $d'=d-\frac{3}{2}''$ and taking the values of I and c for this type of beam from the table, and writing the equation (A) in the form $I=\frac{Mc}{S}$ and putting $M=320\times6=1920$, $S=10000$ (the safe unit-stress for wrought iron):

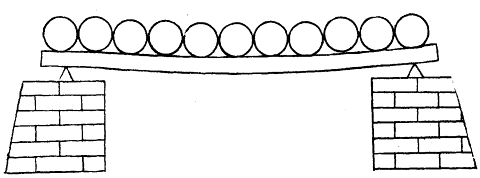
$$\frac{\pi}{64}\left(6d^3-\frac{27}{2}d+\frac{27}{2}d-\frac{81}{16}\right)=\frac{1920\times3\times12d}{2\times10000}$$

and this readily simplifies to

$$96d^3-81d^2-920d-81=0,$$

and from this cubic d must be found. By the methods of higher algebra d is found to lie between 5.1 and 5.2.

A few more practical equations will be added here to show the range of algebraic work needed for the study of applied mechanics. The mechanical meaning of these equations will be fully explained and a careful algebraic study of them will be made.

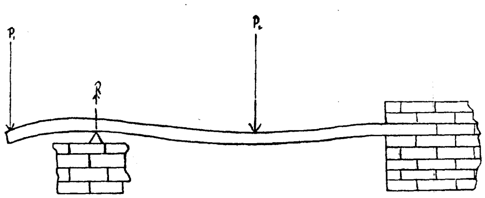


For a simple beam, uniformly loaded, we have:

$V=-wx+\frac{1}{2}wl$ (equation for vertical shear).

$M=\frac{1}{2}wlx-\frac{1}{4}wx^2$ (equation of moments).

$y=\frac{w}{24EI}(-x+2lx-lx)$ (equation of the elastic curve).

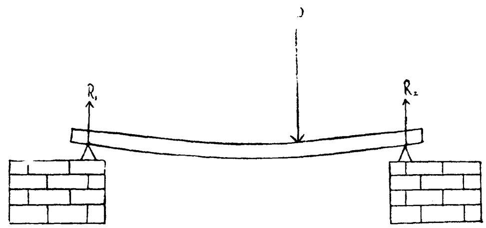


For a simple beam with concentrated loading:

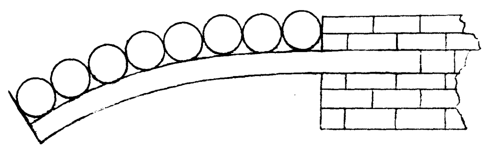
$V=-wx+R_1-\Sigma P$ (equation for vertical shear).

$M=Rx-P_1(x-a)$ (equation of moments).

$y=\frac{P}{48EI}(4x^3-3l^2x)$ (equation of the elastic curve).



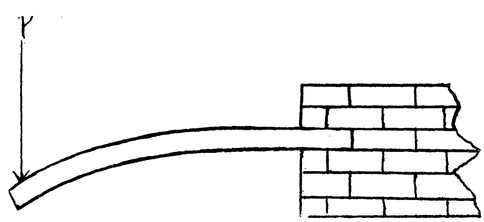
Load distant kl from left end.
$$\begin{cases} y=\frac{1}{EI}(\frac{1}{6}Rx^3+C_1x+C_3), \text{ elastic curve on left of load.} \\ y=\frac{1}{EI}(\frac{1}{6}Rx^3-\frac{1}{6}Px^3+\frac{1}{2}Pklx^2+C_3x+C_4), \text{ for curve on right.} \end{cases}$$



For a cantilever beam, uniformly loaded:

$M=-Px-\frac{1}{2}wx^2$ (equation of moments).

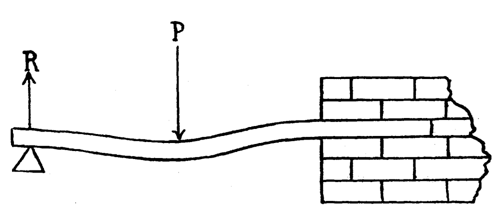
$y=\frac{4wl^3}{24EI}x-\frac{w}{24EI}x^2$ (equation of the elastic curve).

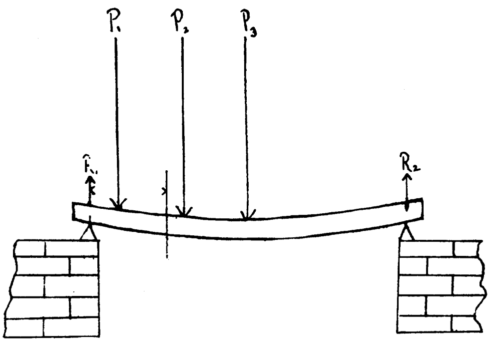


For a cantilever with a single load P:

$y=\frac{P}{6EI}(3l^2x-x^3)$ (equation elastic curve).

Point out which of the foregoing equations are included in the type forms of November outlines and which are cubics and quartics, and give their types.





The following enumeration includes the values of the maximum deflection, Δ, in beams of various types:

For a cantilever loaded at the end with W: $\Delta = \frac{1}{3}\cdot\frac{Wl^3}{EI}$.

For a cantilever loaded uniformly with W: $\Delta = \frac{1}{8}\cdot\frac{Wl^3}{EI}$.

For a simple beam loaded at the middle with W: $\Delta = \frac{1}{48}\cdot\frac{Wl^3}{EI}$.

For a simple beam uniformly loaded with W: $\Delta = \frac{5}{384}\cdot\frac{Wl^3}{EI}$.

For beams to be of uniform strength throughout it is necessary that the cross-sections shall vary with the fiber stress.

For a simple beam loaded at the middle and of constant breadth:

$$d^2 = \frac{3P}{Sb}x \text{ (parabolic profile).}$$

For same with constant depth:

$$b = \frac{3P}{Sd^2}x \text{ (triangular or trapezoidal plan).}$$

Same beam uniformly loaded and constant breadth:

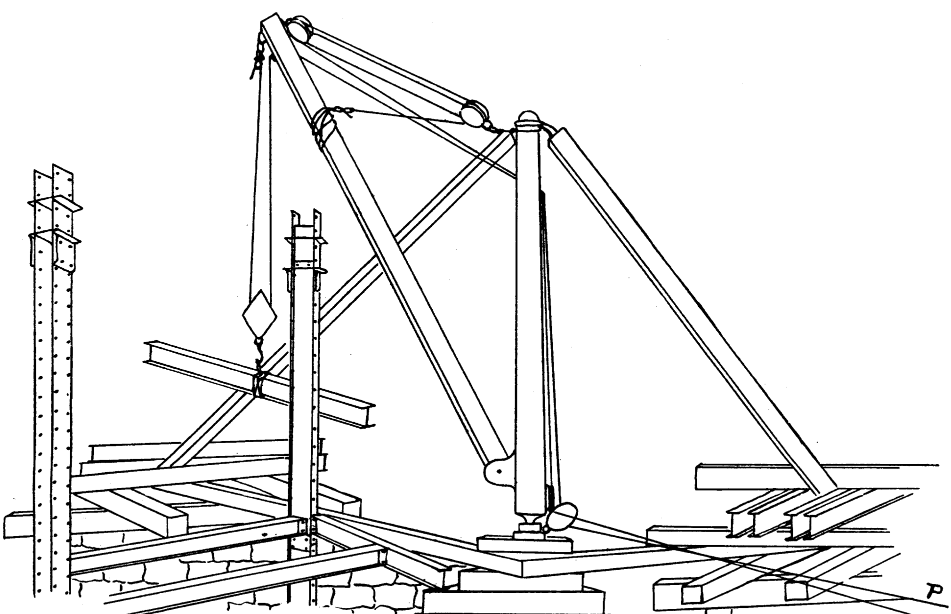
$$d^2 = \frac{3w}{Sb}x \text{ (parabolic profile).}$$

Same beam uniformly loaded and of constant depth:

$$b = \frac{3w}{Sd^2}(lx - x^2 \text{ (parabolic plan).}$$

Now make a full study of fractions, using fractional equations largely. In connection with this study bring out the real motive of greatest common divisor and least common multiple and teach these subjects here. Prove that the familiar method of successive division gives the greatest common divisor.

We shall next develop the theory of exponents and logarithms, the latter immediately after the former.



DERRICK.

Applied Trigonometry

The force P raises, lowers, and supports the boom while Q raises and lowers the load. If the load supported by the derrick in the cut is a 12″ steel I beam, 20′ long, and weighing 50 lbs. to the running foot, what members of the derrick are in tension? In compression? In shear?

Suppose the boom to be 40′ long, the guys to be 42′, and the upright 30′, what are the strains in the ropes, in the boom, and in the guys when the boom is inclined to the upright at an angle of 30°? Of 45°? Of 60°? Of 90°?

Answer the last question (a) when the boom is in the vertical plane of either guy and the upright, (b) when the vertical plane of the boom and upright is midway between the planes of the guys and the upright.

With each of the inclinations of the boom mentioned in the next to the last question, what is the horizontal thrust at the foot of the upright?

What are the forces tending to separate each of the two pulley-blocks at the base of the upright from the upright?

If the boom is of pine and is square in section, what dimensions should it have to carry the load with a factor of safety of 3? (Use for the ultimate compressive strength of pine 8000 lbs. per square inch.)

If the section is 14″ square, what is the factor of safety?

If the maximum load is a 12″ steel I beam, weighing 50 lbs. per running foot, what is the factor of safety?

For the inclinations of the boom to the upright mentioned above what would the strains be if the I beam were replaced by a block of granite 6′×5′×4′? (Granite weighs on the average about 170 lbs. per cubic foot.)

Suppose a green pine tree with dense foliage to be 50′ tall, the butt 18″ in diameter, and the distance from the ground to the lower limbs 8′. Suppose the widest part of the bowl is 14′ in diameter. What will be the moment of the force of a wind blowing against the tree at the rate of 100 miles per hour about a section 2′ above the surface of the ground?

Note.—This velocity, which is the maximum value used by architects, produces a pressure of 50 lbs. per square foot.

What must be the stress in the fibers of the section indicated to resist overturning?

Taking the ultimate tensile strength of green pine to be 12,000 lbs. per square inch, would the tree break off at the section under the force of such a storm?

Numerous trigonometric exercises will be obtained from the exercises at the end of *Applied Mathematics* in the November Course of Study.

Pedagogic School

Algebra for the Seventh and Eighth Grades

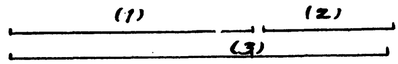


Fig. 1

1. If line 1 is 1 inch long and line 2 is ½ inch long, and when laid down along line 3, as shown in the figure, they reach just as far as line 3, how long is line 3?

Direction: Write your full answer in symbols, thus:

1 inch + ½ inch = 1½ inches.

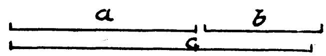


Fig. 2

2. The length of one line is a inches, of another is b inches, and of a third is c inches. When the first two are put on the third, as in the figure, they exactly cover it. How long is c? Write your answer in symbols as in Exercise 1. This is called adding the lines.

How would you subtract a and b?

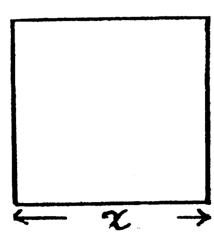


Fig. 3

3. One side of the square is x. How long is a string that will just reach around it?

Remark.—Five times one x is written $5x$ and is read five x.

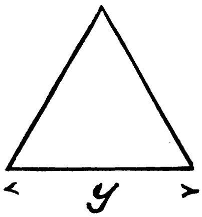


FIG. 4

4. The sides of the triangle are all equal to y (equilateral triangle). How long is a line that will just reach around it ?

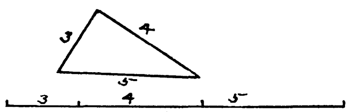


FIG. 5

5. If the sides are as shown and the triangle broken at a corner and straightened out into a line, how long would the line be? Write your answer in symbols.

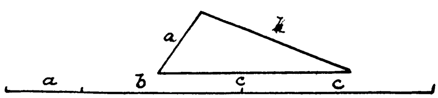


FIG. 6

6. The sides of the triangle are of the lengths shown in the figure. If the triangle were opened at one corner and straightened out into a line, how long would the line be?

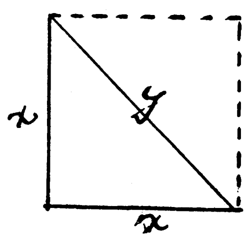


FIG. 7

7. Sides as shown in the figure. How far is it around the triangle?

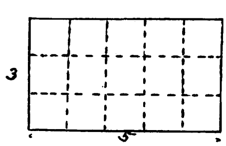


FIG. 8

8. How far is it around the figure?

9. What is the area?

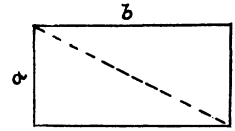


FIG. 9

10. What is the area?

11. What is the area of the oblong? Cut it with a diagonal. What is the area of one part?

REMARK.—The product of two different letters is indicated by writing them side by side. Thus xy means the product of x and y.

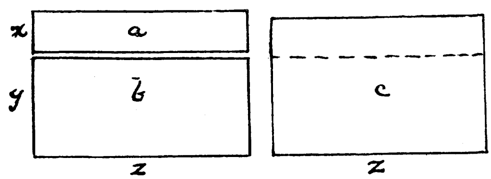


FIG. 10

12. The two oblongs a and b are put together into one as indicated. How high is c?

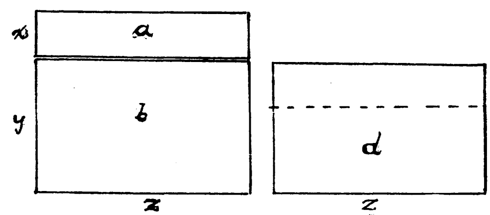


FIG. 11

13. If an oblong as large as a is cut from the oblong b as shown, leaving oblong d, how high is d ?

REMARK.—Remember we indicate the difference of 6 and 2 thus: 6—2.

14. What is the area of a? of b? of c? of d?

REMARK.—5 times x is written $5x$.

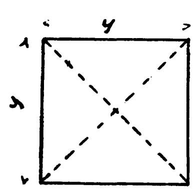


FIG. 12

15. What is the area of the square?

REMARK.—$a \times a$ is written a^2.

16. Cut the square across with a diagonal. What is the area of one part? Of the other part?

17. Cut it across with another diagonal. What is the area of one of the four parts formed?

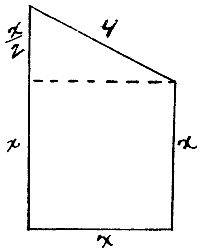


FIG. 13

18. How far is it around this figure? How long is the longest side? What is the area of the figure?

Definition: The expression a/b means the quotient of a divided by b.

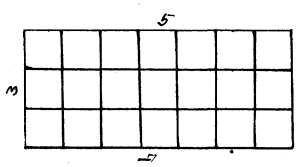


FIG. 14

19. What part of the oblong is one horizontal row of squares? One vertical row? What part of the oblong is one small square?

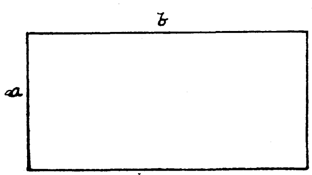


FIG. 15

20. What part of the oblong is one horizontal row of square units? One vertical row? What part is one square unit? How far around the oblong?

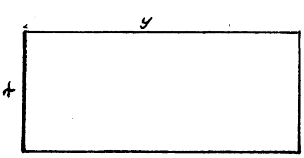


FIG. 16

21. What part of the oblong is one square unit? One horizontal row of them? One vertical row? How far around the oblong?

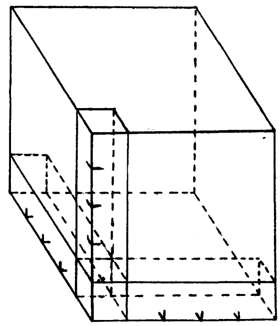


FIG. 17

22. What is the volume of the cube? How many small squares in one face? How many small cubes in one horizontal layer? What part of the large cube is one small cube?

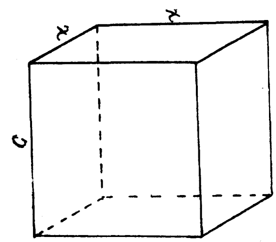


FIG. 18

23. What is the volume? What is the area of a face? What part of the large cube is one small cube?

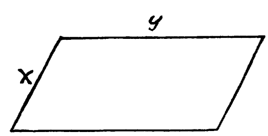


FIG. 19

24. How far is it around the parallelogram? How far is it half way round? One-third of the way round? One ath of the way round?

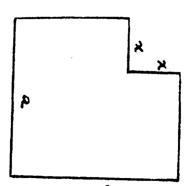


FIG. 20

25. What is the perimeter of the figure?

Definition: Perimeter is the sum of the boundary lines.

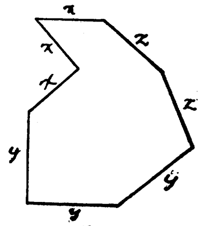


FIG. 21

26. What is the perimeter?

Definition: The parts of your expression which are connected by the plus signs are called the *terms*. They are also called terms when one or more of the signs are minus.

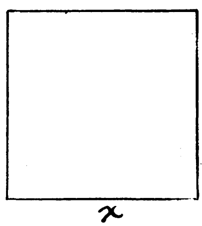


FIG. 22

27. Measure the perimeter of the square with a scale calling $\frac{1}{4}'' = 1'$, and write an equation to find x. What is the value of x? Of $3x$?

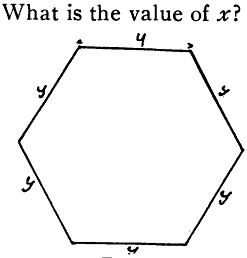


FIG. 23

28. Measure the perimeter and find y.

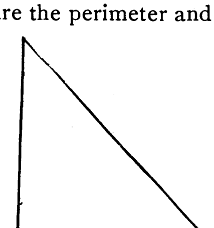


FIG. 24

29. If the area of the right triangle is 32, what is x? (See Exercise 15.)

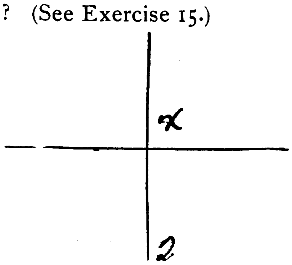


FIG. 2

30. Supposing the two lines to intersect so that all the four angles are equal; denoting one of the angles by x, what will represent the sum of all four?

NOTE.—In this case the lines are said to be *perpendicular* and the angles are right angles.

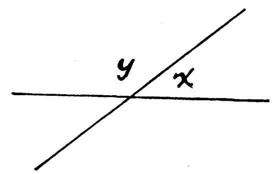


FIG. 26

31. Draw on paper two lines not perpendicular. Cut the paper along the lines and notice what angles will fit over one another. If the angles are designated as in the drawing, what will designate the sum of all four?

What will represent the sum of each pair of equal angles?

What will represent the sum of the two angles below the horizontal line?

What will represent the sum of any two angles lying on the same side of either line?

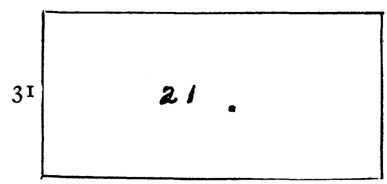


FIG. 27

32. The area of the oblong is 21 square inches and the height is 31 inches. What is the length of the base?

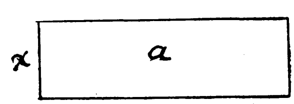


FIG. 28

33. The area of the oblong is a square inches and the height is x. How long is the base?

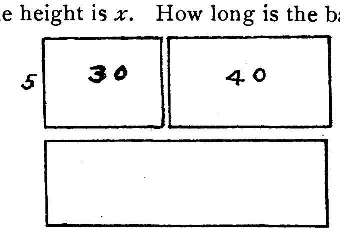


FIG. 29

34. The area of the first oblong is 30 square inches and of the second 40 square inches. Both are 5 inches high. The two are pushed together into one oblong with the same height. How long is the large oblong?

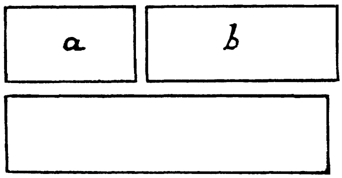


FIG. 30

35. If the area of the first oblong had been a of the second b, and if the altitude of all had been c, how long would the third oblong have been?

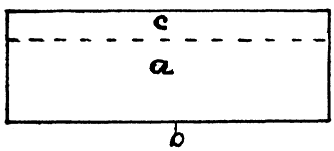


FIG. 31

36. If from an oblong whose base is b inches and whose area is a square inches, a strip is cut off parallel to the top, whose area is c square inches, how high will the oblong be which is left? How broad is the strip?

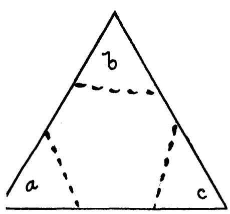


FIG. 32

37. Draw an equilateral triangle. With a protractor measure one of the angles (corners) and fit angle a over angle b. Which is the larger? Write what your experiment proves in symbols. Fit angle a over angle c. How do they compare in size? Symbolize what your exponent shows. Both taken together show what?

REMARK.—Study your protractor and notice its shape; where the center is; what part of a full circle it covers; how it is graduated.

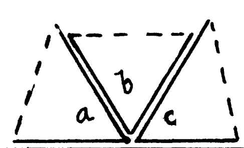


FIG. 33

38. Place the loose corners of the triangle of Example 36 as shown here with their vertices all at a point: how much of the plane around the point will they cover? How many degrees? What equation does this suggest?

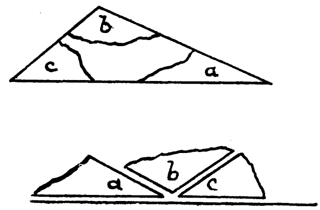


FIG. 34

39. Draw on paper a triangle of any irregular shape and cut, or tear, off the angles. Place these angles around a point o as shown. What part of the plane about the point do they fill? Denoting the angles as in the figure, write an equation for the sum of the angles in degrees.

40. Repeat this operation for several triangles of various shapes. What is the sum of all the angles of any triangle? Letting the angles of any triangle be denoted by a, b and c, what equation can you write for their sum?

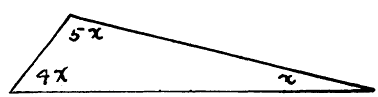


FIG. 35

41. The smallest angle of the triangle is x. If the next larger is four times as large as this and the largest is five times as large, what will denote the other two angles? The sum of all the angles will be denoted by what? To how many degrees is this equal?

42. From what has been said can you tell how many degrees there are in the sum of the two angles of the figure? Call the small angle x. What will represent the large one? If the large one be called y degrees, how many degrees in the small one?

Definition: When the sum of two angles is equal to 180°, each angle is said to be the supplement of the other.

43. What is the supplement of 60°, of 100°, of 80°, of 67°, of y degrees? Of a degrees? Of $x+y$ degrees?

44. Referring to Exercise 40, can you find all three angles of a triangle if the largest is three times the smallest and the second largest is twice the smallest? Denote the smallest angle by x.